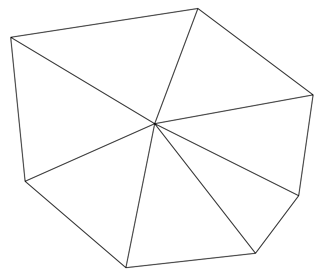
Trigonometry

Trigonometry is the study of triangles. Triangles are of interest because any closed shape this straight sides can be made from triangles.



The word *Trigonometry* comes from two greek words meaning *to measure a triangle*. Trigonometry deals with the relationship between angles and sides of triangles.

- Trigonometric functions based on right triangles.
- Angles measured (convention) in a counter-clockwise direction.
- Directions are measured upward and to the right (positive directions).
- Units: degrees and radians ($360° = 2\pi\, rad$).
- The sum of the angles in a triangle equals $180°$

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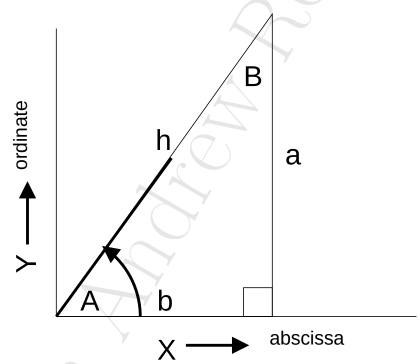


Trigonometric functions

Four basic trigonometric functions defined by ratios of side lengths on a right triangle.

$$\tan(A) = \frac{a}{b} \qquad \sin(A) = \frac{a}{h}$$

functions based on the side opposite the angle divided by another side. The complement or 'co' functions are:

$$\cos(A) = \frac{b}{h} \qquad \cot(A) = \frac{b}{a}$$

Two additional trigonometry functions are:

$$\sec(A) = \frac{h}{b} \qquad \csc(A) = \frac{h}{a}$$

The last relationship needed to characterize a right triangle is:

$$a^2 + b^2 = h^2$$

- Given one trigonometry function, all other trigonometry functions can be determined.
- Given one trigonometric functions and one length of a side, the triangle can be reconstructed.

What is the sin and cos of an angle(A) with a tangent of 2?

$$\frac{a}{b} = 2$$

Because similar triangles have identical trigonometric relationships, any scale can be used: let a=2 and b=1. The hypotenuse then must have a length of $\sqrt{5}$.

$$\sin(A) = \frac{2}{\sqrt{5}}$$
$$\cos(A) = \frac{1}{\sqrt{5}}$$

Everyone should be familiar with some fundamental values for various trigonometric functions, as well as the graphs of these functions, so that the sensibility of answers can be assessed.

angle	tan	sin	cos
0	0	0	1
30	$\frac{1}{\sqrt{3}}$	0.5	$\frac{\sqrt{3}}{2}$
45	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
60	$\sqrt{3}$	$\frac{\sqrt{3}}{2}$	0.5
90	∞	1	0

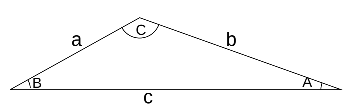
Three less commonly discussed rules in trigonometry can help in solving problems. These rules apply to all triangles.

- Rule of cosines.

$$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos(A)$$
$$b^2 = a^2 + c^2 - 2 \cdot a \cdot c \cdot \cos(B)$$

- Sine rule.

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$



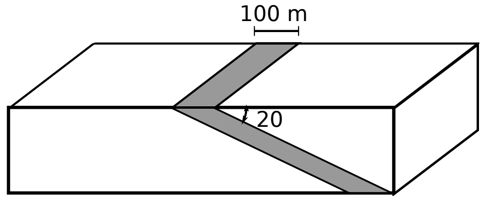
In many instances, the area of a triangle is needed. Two simple equations are used to calculate the area of a triangle. The first is the most common form:

$$Area = \frac{1}{2}(base)(height)$$

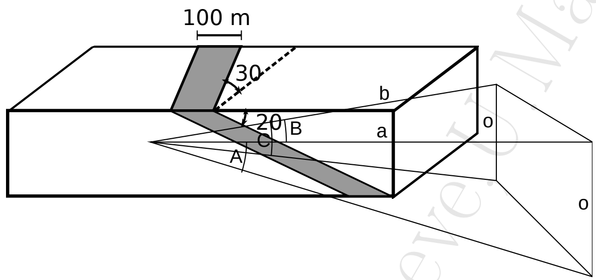
A more obscure equation for calculating the area of a triangles is Heron's Formula:

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$
$$s = \frac{a+b+c}{2}$$

How thick is a unit that is 100 m wide in map view and dips at an angle of 20°?



How thick is a unit that is 100 m wide in map view with an *apparent* dips at an angle of 20°?



A is the apparent dip.

B is the angle needed to rotate direction of apparent dip to true dip direction.

C is the true dip.

$$\tan(A) = \frac{o}{a}$$
$$\cos(B) = \frac{b}{a}$$
$$\tan(C) = \frac{o}{b}$$

Combining these equations:

$$\tan(C) = \frac{\tan(A)}{\cos(B)}$$
$$C = \tan^{-1}\left(\frac{\tan(A)}{\cos(B)}\right)$$

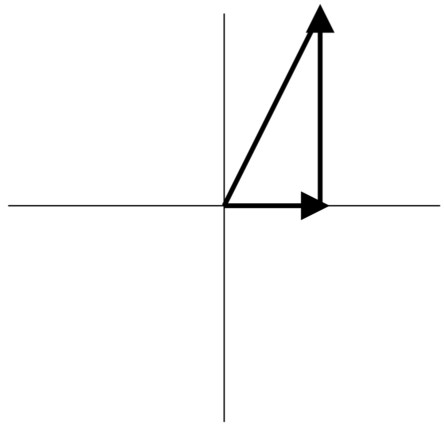
Vectors

Vectors are used to represent directional properties. A vector has a magnitude and a direction. The vector

$$\vec{a} = 1i + 2j$$

can be visualized using Cartesian coordinates as a line starting at (0,0) and ending at (1,2).

The i and j in this equation are unit vectors pointing in the x and y directions. Alternatively, using Polar coordinates, a vector is described with an angle and a distance or radius.



This vector has an x and y component, and the resultant vector is the sum of the two components. Vectors are added by connecting them head to tail, than drawing a line from the beginning and end of the chain of vectors.

This vector can be written in matrix notation as:

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

In Polar coordinates, this is a vector with a length (r) of 2.24 at an angle (θ) of 26.6°.

Going back and forth between the different coordinate systems can be done through some simple trigonometry.

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{x}{y}\right)$$
$$x = r \cdot \cos\theta$$
$$y = r \cdot \sin\theta$$

Triangulation

In field geology, outcrops and other geologic features are located through triangulation.

- Two or more objects sighted with compass.
- Lines with the compass bearings are drawn passing through the known objects,
- Intersecting point is position sighted from.

Fine for one location, but one might make observations and write down the bearings to known objects, then solve locations at end of day. One may not want to graphically reconstruct dozens of observations. How can the locations be calculated without going through the graphical reconstruction?

Given the angles of two objects with known position, some simple trigonometry can be used to calculate the position of an object.

Vacher suggests making equations base on a unit vector:

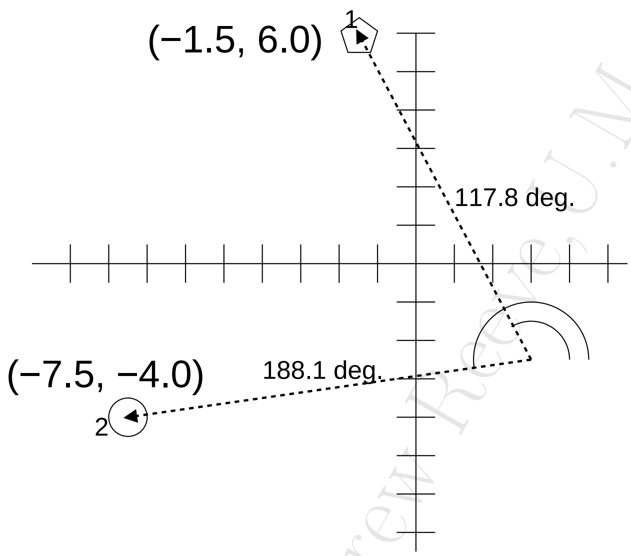
$$\vec{v_1} = \cos(\theta_1)\vec{i} + \sin(\theta_1)\vec{j}$$
$$\vec{v_2} = \cos(\theta_2)\vec{i} + \sin(\theta_2)\vec{j}$$
$$\tan(\theta_1) = \left(\frac{y_1 - y_?}{x_1 - x_?}\right)$$
$$\tan(\theta_2) = \left(\frac{y_2 - y_?}{x_2 - x_?}\right)$$

rearranging the equations based on the tangents:

$$\tan(\theta_1)(x_1 - x_?) = (y_1 - y_?)$$
$$x_1 \cdot \tan(\theta_1) - x_? \cdot \tan(\theta_1) = y_1 - y_?$$
$$x_1 \cdot \tan(\theta_1) - y_1 = x_? \cdot \tan(\theta_1) - y_?$$

similarly

$$x_2 \cdot \tan(\theta_2) - y_2 = x_? \cdot \tan(\theta_2) - y_?$$



Two simultaneous equations can be made from this data:

$$(-1.5) \cdot \tan(117.8) - (6.0) = x_? \cdot \tan(117.8) - y_?$$
$$(-7.5) \cdot \tan(188.1) - (-5) = x_? \cdot \tan(188.1) - y_?$$

$$-3.155 = x_? \cdot -1.897 - y_?$$
$$2.933 = x_? \cdot 0.142 - y_?$$

$$-6.088 = x_? \cdot -2.039$$
$$2.986 = x_?$$
$$-3.280 = y_?$$

Vectors can be manipulated in a variety of ways. Vector addition has already been shown. Vector subtraction can be handled in a variety of ways. Vectors can be placed head to head (order is important), with the subtracted vectors head placed next to the original vectors head. Perhaps a less confusing and more consistent way to do this is to treat subtraction as two operations, addition and scalar multiplication.

$$\vec{v_1} - vecv_2 = \vec{v_1} + (-1) \cdot vecv_2$$

Vector multiplication can be done in two ways, and these methods have a variety of names.

The dot product (or inner product) is the sum of the products of each vector component. This can be considered as multiplying a row vector and a column vector:

$$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} = scalar$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 4$$

In linear algebra, the dot product is related to the angle between two vectors, and is used to determine if vectors are orthogonal. If two unit vectors at right angles are chosen, they will have the form

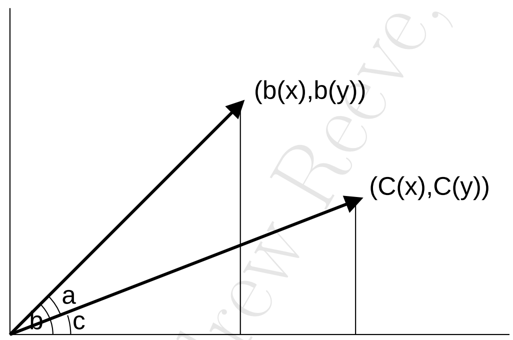
$$\begin{bmatrix} cos(\theta) \\ sin(\theta) \end{bmatrix} and \begin{bmatrix} -sin(\theta) \\ cos(\theta) \end{bmatrix}$$

The inner product of these vectors is:

$$-sin(\theta) \cdot cos(\theta) + sin(\theta) \cdot cos(\theta) = 0$$

demonstrating that when two vectors are at right angles, their inner product will be zero.

When vectors are not orthogonal, the inner product is related to the angle between the vectors.



using the identity:

$$cos(a) = cos(b - c) = cos(b)cos(c) + sin(c)sin(b)$$

$$cos(b) = \frac{b_x}{\sqrt{b_x^2 + b_y^2}}$$

$$sin(b) = \frac{b_y}{\sqrt{b_x^2 + b_y^2}}$$

$$cos(a) = \frac{b_x}{\sqrt{b_x^2 + b_y^2}} \frac{c_x}{\sqrt{c_x^2 + c_y^2}} + \frac{b_y}{\sqrt{b_x^2 + b_y^2}} \frac{c_y}{\sqrt{c_x^2 + c_y^2}}$$

$$cos(a) = \frac{\vec{b}^T \vec{c}}{||b|| \, ||c||}$$

This is analogous to the law of cosines.

The outer product or cross product calculates a vector orthogonal to the original vectors. In two dimensions, this is a trivial solution, as the vector will always be oriented in the 'z direction'. In three dimensions, this becomes

$$\vec{a} \times \vec{b} = vector$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \vec{i}(a_y b_z - a_z b_y) + \vec{j}(a_z b_x - a_x b_z) + \vec{k}(a_x b_y - a_y b_x)$$

The length (or norm) or the cross product is also related to a trigonometry function and is the area of a parallelogram with sides corresponding to the vectors:

$$\left|\vec{a} \times \vec{b}\right| = sin(\theta) \left\|a\right\| \left\|b\right\|$$