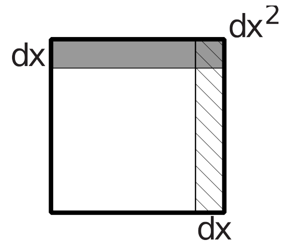
Intro to Calculus

- Calculus is a branch of mathematics that deals with small quantities; quantities that are so small they might be considered insignificant.
- In calculus, degrees of smallness are considered. Consider dx to be the first degree of smallness and dx^2 to be the second degree of smallness. If dx is very small, then dx^2 is negligible.
- Consider a square where dx is a tiny portion of one side. If dx is very small, then dx^2 is negligible.



Rates and Differences

- In calculus, these small quantities are used to express rates, with quantities grouped into constants and variables. If two variables depend on each other, then their ratio expresses a rate.
  - distance and time: speed
  - vertical and horizontal distance: slope
  - volume and time: discharge

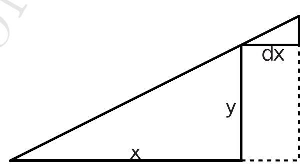
derivatives

- Assume x and y are two variables that depend on each other. If x changes then y must also change because there is a relationship between them.

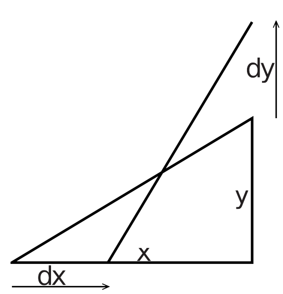
$$x + dx = y + dy \tag{1}$$

- This relationship can be shown using a triangle, with the length of the legs of the triangle proportional to the size of each variable. In this case (fixed rate):

$$\frac{x}{y} = \frac{dx}{dy} \tag{2}$$



Derivatives



- In the above figure, the length of the hypotenuse stays the same as the length of the triangles legs are changed (legs of second triangle not shown).
- In this case, if x decreases y must increase.

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Derivatives

In the case of two triangles with the same hypotenuse, $x - dx = y + dy$

$$h^2 = x^2 + y^2$$
$$h^2 = (x - dx)^2 + (y + dy)^2$$

When differentials are used, a relationship between variables is implied, with dx and dy being negligibly small values.

Derivatives

- When considering a derivative, a relationship between two variables is needed. What is $\frac{dy}{dx}$ with respect to the following equation?

$$y = x^2$$

- If a small amount is added to y, there will be a small change in x.

$$y + dy = (x + dx)^2$$
$$y + dy = x^2 + 2xdx + dx^2$$

Rules of Differentiation

- dx is of the 'first order of smallness' and dx^2 is of the 'second order of smallness'. In this case dx^2 is much smaller than dx and will be considered unimportant.

$$y + dy = x^2 + 2xdx$$

- Subtracting the original equation from this approximation:

$$dy = 2xdx$$
$$\frac{dy}{dx} = 2x$$

Rules of Differentiation

- Consider a cubic equation:

$$y = x^3$$

- Proceeding as before:

$$d + dy = (x + dx)^3$$
$$y + dx = x^3 + 2x^2dx + xdx^2 + x^2dx + 2xdx^2 + dx^3$$
$$y + dy = x^3 + 3x^2dx$$
$$\frac{dy}{dx} = 3x^2$$

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Rules of Differentiation

- If we do this for any simple polynomial, a pattern will emerge:

$$y = x^n$$
$$\frac{dy}{dx} = nx^{n-1}$$

Rules of Differentiation

- Consider an equation with a constant:

$$y = x^2 + 5$$

- Going through the same steps again:

$$y + dy = (x + dx)^2 + 5$$
$$y + dy = x^2 + 2xdx + dx^2 + 5$$
$$\frac{dy}{dx} = 2x$$

- the constant drops out of the equation

Rules of Differentiation

Multiplying by a constant:

$$y = 7x^2$$
$$y + dy = 7(x + dx)^2$$
$$y + dy = 7(x^2 + 2xdx + dx^2)$$
$$dy = 7 \cdot 2xdx$$
$$\frac{dy}{dx} = 14x$$

Rules of Differentiation

- Differentiating a summation is done sequentially.

$$y = 2x + 3x^2$$
$$u = 2x$$
$$v = 3x^2$$
$$y = u + v$$
$$y + dy = u + du + v + dv$$
$$dy = du + dv$$
$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

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Product Rule

Differentiating a product:

$$y = (2x + 5)(x^2 + 7) \tag{3}$$

Could be approached by multiplying out and differentiating sequentially. Alternatively, substitution can be used:

$$\begin{aligned} u &= 2x + 5 \\ v &= x^2 + 7 \\ y &= uv \\ y + dy &= (u + du)(v + dv) \\ y + dy &= uv + vdu + udv + dudv \\ dudv &\approx 0 \\ dy &= vdu + udv \end{aligned}$$

Product Rule

Checking this new equation using $y = (2x + 5)(x^2 + 7)$:

$$\begin{aligned} y &= 2x^3 + 5x^2 + 14x + 35 \\ dy &= 6x^2 + 10x + 14 \end{aligned}$$

Using the product rule:

$$\begin{aligned} y &= (2x + 5)(x^2 + 7) \\ dy &= (2x + 5)d(x^2 + 7) + (x^2 + 7)d(2x + 5) \\ dy &= (2x + 5)(2x) + (x^2 + 7)(2) \\ dy &= 4x^2 + 10x + 2x^2 + 14 \\ dy &= 6x^2 + 10x + 14 \end{aligned}$$

Quotient Rule

Similar methodology can be applied to quotients:

$$\begin{aligned} y &= \frac{u}{v} \\ y + dy &= \frac{u + du}{v + dv} \end{aligned}$$

Perform algebraic division:

$$\begin{aligned} y + dy &= \frac{u}{v} + \frac{du}{v} - \frac{u}{v^2}dv \\ y &= \frac{u}{v} \\ dy &= \frac{vdu - udv}{v^2} \end{aligned}$$

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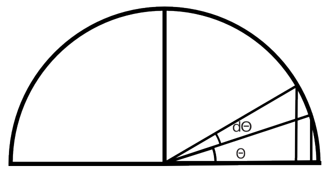
Trig. Rules

Simple analysis of trigonometry functions:



$$y = sin(\theta)$$
$$y + dy = sin(\theta + d\theta)$$
$$dy = sin(\theta + d\theta) - sin(\theta)$$

Trig.Rules

Using the trig. identity:

$$sin(\alpha) - sin(\beta) = 2cos\left(\frac{\alpha+\beta}{2}\right) \cdot sin\left(\frac{\alpha-\beta}{2}\right)$$
$$dy = 2cos\left(\frac{\theta + d\theta + \theta}{2}\right) \cdot sin\left(\frac{\theta + d\theta - \theta}{2}\right)$$
$$dy = 2cos\left(\theta + \frac{d\theta}{2}\right) \cdot sin\left(\frac{d\theta}{2}\right)$$

Trig. Rule

assume $d\theta$ is small and using sin law:

$$dy = 2cos\left(\theta + \frac{d\theta}{2}\right) \cdot \left(\frac{d\theta}{2}\right)$$
$$cos\left(\theta + \frac{d\theta}{2}\right) d\theta$$

assume $\theta + d\theta \approx \theta$

$$dy = cos(\theta)d\theta \tag{4}$$

Exponential

To differentiate the exponential function, start with the infinite series:

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \dots$$
$$y = e^x$$
$$dy = 0 + dx + \frac{2xdx}{2!} + \frac{3x^2dx}{3!} + \dots$$
$$dy = e^x dx$$
$$\frac{dy}{dx} = e^x$$

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