Differentiation Summary

We have now worked through the basic methods for differentiating an equation.

$$y = x^n \quad \frac{dy}{dx} = n \cdot x^{n-1}$$
$$y = e^x \quad \frac{dy}{dx} = e^x$$
$$y = sin(x) \quad \frac{dy}{dx} = cos(x)$$
$$y = cos(x) \quad \frac{dy}{dx} = -sin(x)$$
$$y = ln(x) \quad \frac{dy}{dx} = \frac{1}{x}$$

In addition, we have several important rules. If u and v are both functions of x then:

- Sum Rule

$$y = u + v$$
$$dy = du + dv$$

- Product Rule

$$y = u \cdot v$$
$$dy = vdu + udv$$

- Quotient Rule

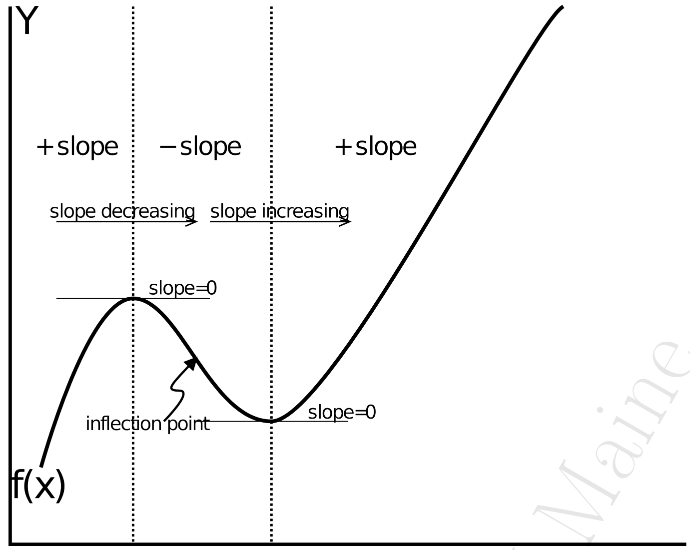
$$y = \frac{u}{v}$$
$$dy = \frac{vdu - udv}{v^2}$$

- Chain Rule

$$\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du}$$

Maximum and Minimum

- One application of calculus is optimization
- seek the maximum or minimum in some situation.
- Examples: finding the minimum cost for building something, maximizing the amount of contamination removed from ground water, and minimizing the error in a problem.
- The maximum or minimum will be a peak or trough in a function.
- The tangent line at this point, and therefore the derivative, will be zero.



- slope decreases with x around a maximum, and increases with x around minimum.
- The slope (first derivative) plotted against x, gives another function that can again be differentiated (second derivative).

$$\frac{dy}{dx} = slope$$
$$\frac{d^2y}{dx^2} = \frac{d(slope)}{dx}$$

- When $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$, a minimum is found.
- When $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$, a maximum is found.
- What if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$? What kind of equation might give this result?

Partial Derivatives

- When dealing with functions containing two (or more) variables, differentiation becomes more complex.
- Instead of dealing with the slope on an x-y graph, we need to deal with a function that varies in different directions.
- Elevation on a topographic map is a function of x and y position. We can visualize a function with two variables as a surface and calculate directional slopes. These directional slopes are the partial derivatives of the function.

Linear Regression from Calculus

- Linear regression relies on these principles.
- This problem involved minimizing the sum of the squared residuals: Summing the squared residuals will produce a parabolic function containing one point with a slope of zero.
- seek the slope and intercept that minimize the sum of the squared residuals.
- The residuals are then a function of two variables and we need to rely on partial derivatives to find the lowest point on a plot of residuals vs. m and b

Linear Regression from Calculus

$$Res = y_{data} - y_{calc}$$
$$Res = y_{data} - (m \cdot x + b)$$

$$Res^2 = y_i^2 - 2 \cdot y_i(mx_i + b) + (mx_i + b)^2$$
$$Res^2 = y_i^2 - 2y_i mx_i - 2y_i b + m^2 x_i^2 + 2bmx_i + b^2$$

y_i is sum of y values, y_i^2 is sum of squared values, etc.

Linear Regression from Calculus

Taking the partial derivative with respect to m and b:

$$\frac{\partial\left(Res^2\right)}{\partial b} = -2y_i + 2mx_i + 2 \cdot n \cdot b$$
$$\frac{\partial\left(Res^2\right)}{\partial m} = -2y_i x_i + 2mx_i^2 + 2bx_i$$

n is number of values in X of Y, comes from $\Sigma b = n \cdot b$

When m and b make both equations equal zero, the equation that minimizes the residuals is determined. Note that no maximum exists for a parabolic function.

Linear Regression with Python

```
import numpy as np
X=np.arange(1,10)
Y=X+np.random.uniform(-1,1,9)#add random numbers to line w/ slope of 1
X1=np.sum(X)
X2=np.sum(X**2)
Y1=np.sum(Y)
XY=np.sum(X*Y)
```

$$Y = m \cdot X + n \cdot b$$
$$X \cdot Y = m \cdot X^2 + b \cdot X$$

$$43.296 = 9 \cdot b + m \cdot 45.$$
$$277.33 = b \cdot 45 + m \cdot 285$$

Linear Regression with Python

The can be solved by substitution or using some linear algebra:

$$\begin{bmatrix} X & n \\ X^2 & X \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \Sigma Y \\ \Sigma X \cdot Y \end{bmatrix}$$

```
import numpy as np
from numpy import linalg as la #access linear algebra module
X=np.arange(1,10) #make a vector
Y=X+np.random.uniform(-1,1,9) # add 'noise' to same vector
X1=np.vstack((np.ones(9),X))
A=np.dot(X1,np.transpose(X1))
B=np.array([np.sum(Y),np.sum(X*Y)])
b,m=la.solve(A,B) #find intercept and slope
```

Linear Regression with Python

- could also calculate correlation coef.
- $r = \frac{(X-X_{ave})\cdot(Y-Y_{ave})}{\sqrt{\Sigma(X-X_{ave})^2}\sqrt{\Sigma(Y-Y_{ave})^2}}$

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```

Integration

- Integration is similar to summation.
- Summation involves adding together finite quatities (although this definition looses some of its meaning in infinite series).
- Integration involves adding together and infinite number of infinitesimal pieces.

  The symbols for summation and integration are:

$$\sum_{x=0}^{\infty} \frac{1}{2^x}$$

$$\int_0^{\infty} \frac{1}{2^x} dx$$

- 0 and ∞ are the limits of integration (or summation). They represent the range of values for x.
  - When included, the function is a definite integral
  - when excluded, the function is an indefinite integral.

Integration

A simple integral is:

$$\int (0 \cdot x + 1)dx = x + C$$

The states that the sum of all the little pieces of x equal x plus a constant. Where does the constant come from?

Integration

We can see where it comes from by turning this equation around:

$$y = x + C \quad (1)$$

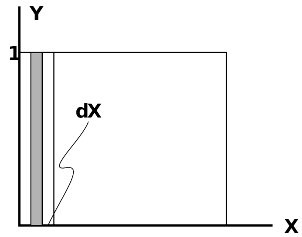
$$dy = dx \quad (2)$$

$$\quad (3)$$

The constant disappears when we differentiate to find how y varies with x. This constant does not affect the rate of change, or slope of the line (m=0), it only shifts the position of the line.

Integration

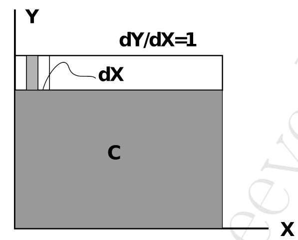
When integrating a function, we sum all the pieces, in this case the little rectangles shown below.



The sum of these is simply the area of a rectangle with a height of 1, or $1 \cdot x$.

Integration

Because our equation only provides information about the slope of the function and not the intercept, a constant needs to be included in the solution. To figure out what the constant is, we need additional information.



$$\int dx = 1 \cdot x + C$$

In this particular case, C=0 and we find the area of a rectangle $(b \cdot x)$.

Integration

Looking at a slightly more complicated case, with a line that increases in slope with x:

$$\frac{dy}{dx} = x$$
$$dy = xdx$$

If x was a constant:

$$\int xdx = x \int dx$$
$$x \int dx = x^2$$

This is clearly an over-simplification

Integration

Using the average value for each x to approximate it, $\frac{x}{2}$.

$$\int xdx \approx \frac{1}{2}x^2 + C$$

We've played fast and loose with the math, however, we arrive at a good (exact!) approximation for the integral while illustrating what integration is about. We can also see, at least in this case, that integration and differentiation are opposite functions.

Areas under curves can be found using integration. If we wish to find the area under the curve $y = x$, the area is unbounded. Limits need to be imposed on the function and this is done by listing the starting and ending points at the base and top of the integral sign.

$$\int_0^5 x dx$$

Knowing that the integral is the opposite of differentiation, we seek the function that, when differentiated, gives x.

$$\int_0^5 x dx = \left[\frac{1}{2}x^2\right]_0^5$$

$$\int_0^5 x dx = \frac{1}{2}5^2 - \frac{1}{2}0^2$$

$$\int_0^5 x dx = \frac{25}{2}$$

Just as with differentiation, integration has several useful rules for solving more complex functions. Many of these follow directly from the differentiation rules. One of the more useful tricks is integration by parts, which follows directly from the product rule.

$$y = u \cdot v$$

$$dy = udv + vdu$$

$$\int dy = \int udv + \int vdu$$

$$y = \int udv + \int vdu$$

$$u \cdot v = \int udv + \int vdu$$

$$\int udv = u \cdot v - \int vdu$$

The final formula is the equation for integration by parts.

Substitution can be used to simplify a problem.

$$\int \sqrt{x+3}\, dx$$

$$u = x + 3$$

$$du = dx$$

$$\int \sqrt{u} du = \frac{2}{3}u^{\frac{3}{2}}$$

$$\int \sqrt{x+3} dx = \frac{2}{3}(x+3)^{\frac{3}{2}}$$

Using Sympy

- What is 'Sympy'?
  - python module for symbolic mathematics
  - allows equations to be expressed as symbols
  - isympy, custom coupling of sympy and ipython
- Assigning variables
  - import sympy (or use isympy)
  - 'true' division: from __future __ import division
  - create symbols/variables (sympy.var, sympy.Symbol)
    - var creates global, Symbol creates local variables
- types in sympy: Rational, Real, Integer
- special numbers 'pi','oo'

Sympy Algebra 1

```
##executed by 'isympy'
from __future__ import division
from sympy import *
x, y, z = symbols('xyz')
k, m, n = symbols('kmn', integer=True)
f, g, h = map(Function, 'fgh')
##################
Rational(2**3,6**9)
Integer(2**3/6**9)
Real(2**3/6**9)
N(sqrt(2))
N(sqrt(2),50)
oo # 'double ohs' for inf
1+pi
x,y,z=var('x,y,z') # note string passed to 'var'
```

Sympy Algebra 2

```
f=(x+y)**2
f.expand()
f=2*x*y+x**2+y**2
factor(f)
f=x**2+x**2*y
collect(f,x) # separate like var
f=sin(x)**2+cos(x)**2
trigsimp(f)
f=cos(x+y)
expand_trig(f)
(solve(x**2+5))
```

Sympy:Summation and series

```
sum(.5**(x), (x, 0, oo))
sum(Rational(2,3)**x,(x,0,oo))
sum(2**(x), (x, 0, oo))
series(exp(x),x,0,5)
exp(x).series(x,0,9)
cos(x).series(x,0,12)
```

Sympy:Differentiation

```
diff(tan(x),x)
diff(cos(x),x,1)
diff(cos(x),x,2)
diff(cos(x)/sin(2*x),x)
##check this
limit((cos(x+y)/sin(2*x+y)-cos(x)/sin(2*x))/y,y,0)
##another check
N(diff(cos(x)/sin(2*x),x).subs(x,1))
f=(cos(x+y)/sin(2*(x+y))-cos(x)/sin(2*x))/y
N(f.subs(x,1).subs(y,.001))
```

Sympy:Integration

```
integrate(sqrt(x,x)
integrate(exp(x),x)
integrate(1/x,(x))
integrate(exp(x),(x,1,oo))
```