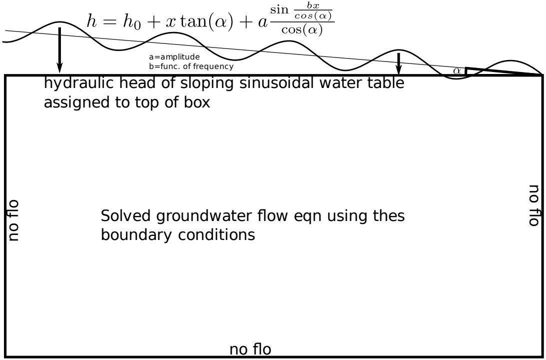
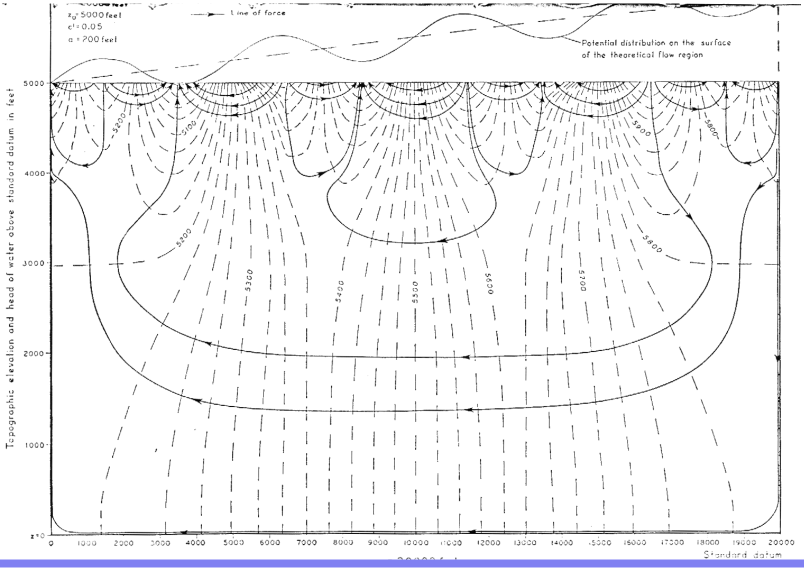
Types of Groundwater Models

- Analog (eg. electrical,sandbox)
  - difficult to modify
  - difficult to make multilayer systems
- Analytic
- Numerical

Analytic Models

- Classic Paper by Toth (1963)
- Solved groundwater flow equation for idealized basin





z_0 = 5000 feet
c' = 0.05
a = 200 feet
Line of force
Potential distribution on the surface of the theoretical flow region
Topographic elevation and head of water above standard datum in feet
5000
4000
3000
2000
1000
z = 0
5200
5100
5900
5800
5200
5300
5400
5500
5600
5700
5800
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
11000
12000
13000
14000
15000
16000
17000
18000
19000
20000
Standard datum

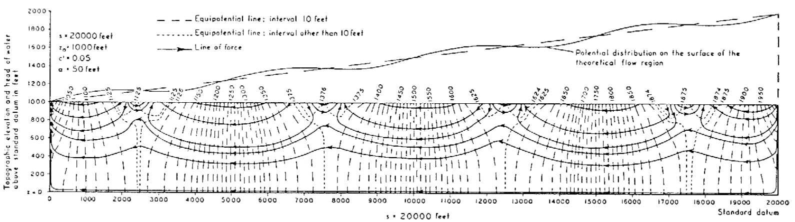


Fig. 2*c*. Potential distribution and flow pattern as obtained by equation 6.

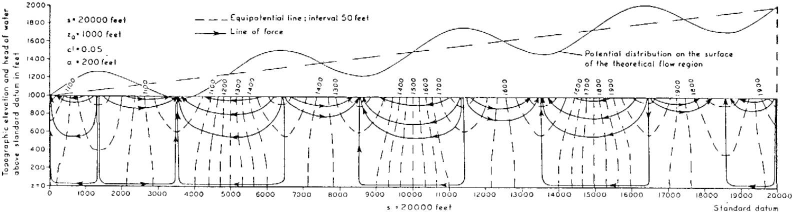
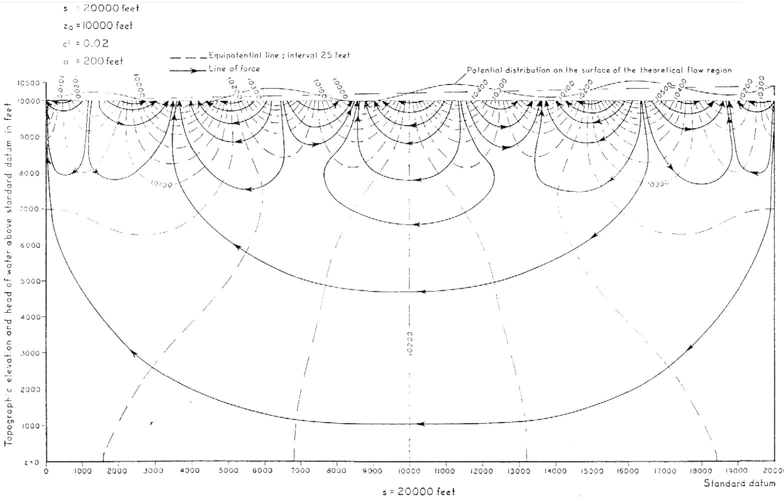


Fig. 2*d*. Potential distribution and flow pattern as obtained by equation 6.


s = 20000 feet
z₀ = 10000 feet
c' = 0.02
a = 200 feet
Equipotential line ; interval 25 feet
Line of force
Potential distribution on the surface of the theoretical flow region
Topographic elevation and head of water above standard datum in feet
Standard datum
s = 20000 feet

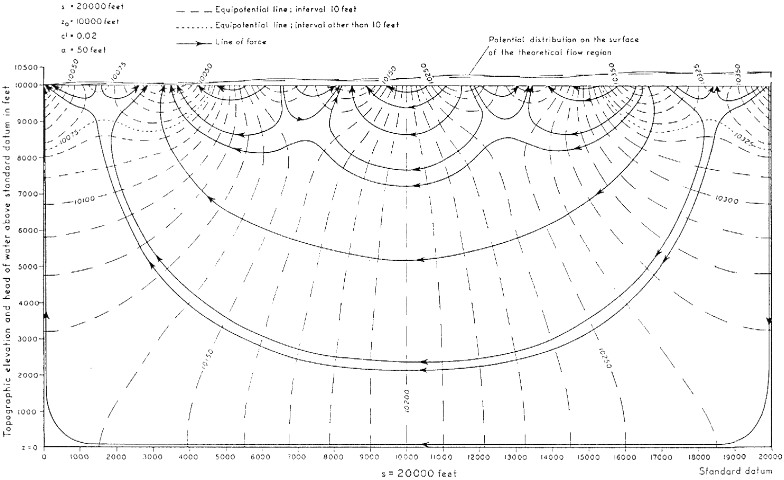


Fig. 2f. Potential distribution and flow pattern as obtained by equation 6.

Current ‘Types’ of Computer Models

Numerical Methods

- Finite Difference
- Finite Element
- Finite Volume
- Analytic Element

Selecting a Model

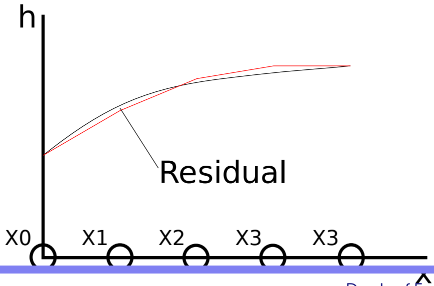
- ‘Canned’ Software
  - Modflow
  - Topodrive
  - Many others
- Program your own
  - Spreadsheet models
  - Software libraries
  - High level languages

Finite Element Method

- Using weighting function (N) to interpolate across elements

$$\hat{h} = h_i \cdot N$$

$$\hat{h}_{01} = h_o \frac{x - x_o}{\Delta x} + h_1 \left(1 - \frac{x - x_o}{\Delta x}\right)$$



$$\frac{\partial^2 h}{\partial x^2} \rightarrow \frac{\partial^2 \hat{h}}{\partial x^2}$$

$$\frac{\partial^2 \hat{h}}{\partial x^2} = R$$

$$\int N \frac{\partial^2 \hat{h}}{\partial x^2} dx = \int N \cdot R dx = 0$$

integrate by parts

$$\sum N \frac{\partial h}{\partial x} - \sum h_i \int \frac{\partial N}{\partial x} \frac{\partial N}{\partial x} dx = 0$$

For each element:

$$\begin{bmatrix} \int_x \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} & \int_x \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x} \\ \int_x \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x} & \int_x \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} N_1 \frac{\partial h}{\partial x} \\ N_2 \frac{\partial h}{\partial x} \end{bmatrix}$$

note that $\frac{\partial N}{\partial x} = \frac{1}{\Delta x}$ is a constant.
numerically integrate

$$\frac{1}{\Delta x_i} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} N_1 \frac{\partial h}{\partial x} \\ N_2 \frac{\partial h}{\partial x} \end{bmatrix}$$

left side applies to fluxes across boundary

Combining the terms from two elements:

$$\begin{bmatrix} \frac{1}{\Delta x} & -\frac{1}{\Delta x} & 0 \\ -\frac{1}{\Delta x} & \frac{1+1}{\Delta x} & -\frac{1}{\Delta x} \\ 0 & -\frac{1}{\Delta x} & \frac{1}{\Delta x} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial h}{\partial x_1} \\ 0 \\ \frac{\partial h}{\partial x_3} \end{bmatrix} \tag{1}$$

Finite Difference Method

- truncated Taylor series used to approximate terms in PDE

$$h(x_0 + \Delta x) = \sum_{n=0}^{\infty} \frac{d(h(x_0))}{dx} \frac{\Delta x^n}{n!}$$

$$h(x_0 + \Delta x) = h(x_0) + \frac{dh(x_0)}{1!dx}(\Delta x) + \frac{d^2 h(x_0)}{2!dx^2}(\Delta x)^2 \cdots$$

$\frac{\partial h}{\partial t}$ can be approximated by truncating the Taylor series after the first term:

$$h(x+\Delta x) = h(x) + \frac{1}{1!}\left(\Delta x \frac{\partial h(x)}{\partial x}\right) + \left(\frac{\Delta x^2}{2}\frac{\partial^2 h(x)}{\partial x^2}\right)\cdots$$

$$\frac{\partial h}{\partial x} = \frac{h(x+\Delta x) - h(x)}{\Delta x} + \left(\frac{\Delta x}{2}\frac{\partial^2 h(x)}{\partial x^2}\right)\cdots$$

Truncating the after the second term results in an error proportional to Δx (first order). When Δx is small, the error will be small.

The forward taylor series approximation is:

$$\frac{\partial h}{\partial x} \approx \frac{h(x+\Delta x) - h(x)}{\Delta x}$$

As similar procedure is followed to develop the backwards Taylor series approximation:

$$\frac{\partial h}{\partial x} \approx \frac{h(x) - h(x-\Delta x)}{\Delta x}$$

$$h(x+\Delta x) = h(x) + \frac{\Delta x}{1}\frac{\partial h(x)}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 h(x)}{\partial x^2} + \frac{\Delta x^3}{3!}\frac{\partial^3 h(x)}{\partial x^3}\cdots$$

$$h(x-\Delta x) = h(x) - \frac{\Delta x}{1}\frac{\partial h(x)}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 h(x)}{\partial x^2} - \frac{\Delta x^3}{3!}\frac{\partial^3 h(x)}{\partial x^3}\cdots$$

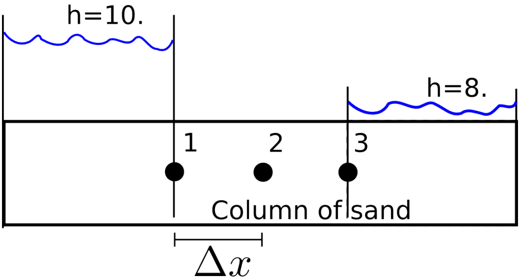
Adding these together yields:

$$h(x+\Delta x) + h(x-\Delta x) = 2h(x) + 2\frac{\Delta x^2}{2}\frac{\partial^2 h(x)}{\partial x^2} + 2\frac{\Delta x^4}{4!}\frac{\partial^4 h(x)}{\partial x^4}\cdots$$

$$\frac{h(x+\Delta x) - 2h(x) + h(x-\Delta x)}{\Delta x^2} = \frac{\partial^2 h(x)}{\partial x^2} + 2\frac{\Delta x^2}{4!}\frac{\partial^4 h(x)}{\partial x^4}\cdots$$

Truncating the higher order terms gives second order accurate approximation for the second derivative. What happens if we subtract above equations?

One-dimension problem I



One-dimension problem II

For a 1-D, homogeneous, isotropic ground-water flow system:

$$0 = \frac{\partial^2 h}{\partial x^2}$$

Applying the finite-difference approximation to this problem results in an algebraic equation of each node in the problem area.

$$h_1 = 10$$
$$\frac{h_1 - 2h_2 + h_3}{\Delta x^2} = 0$$
$$h_3 = 8$$

What is h_2? Note that the finite-difference method solves the problem at discrete points typically spaced at equal intervals.

Selecting a modeling platform

- User needs
  - Ease of use
  - Versatility
  - Cost
- Software sources
  - USGS (MODFLOW, SUTRA and others)
  - USEPA (WHAEM, MODFLOW Support)
  - Commercial (FEFLOW, MIKE SHE, PORFLOW)
  - General CFD packages (OpenFoam, FiPy,...)

Modflow

- Finite Volume (Integrated FD)
- Modular Structure
- Free from USGS
- Commercialized versions, GUI

Streamlines in flownets

- Streamline position can be calculated using the stream function.
- In an isotropic system with an incompressible fluid,

$$\frac{dx}{q_x} = \frac{dy}{q_y} \tag{2}$$

- relates the length of a vector to the discharge.
- Ψ is the stream function, a parameter that has a constant value along a steamline.
- If true, change in Ψ with distance along a streamline must be zero:

$$\frac{d\Psi}{ds} = 0 = \frac{\partial \Psi}{\partial x}\frac{dx}{ds} + \frac{\partial \Psi}{\partial y}\frac{dy}{ds} \tag{3}$$

Combining these last two equations:

$$\frac{\partial \Psi}{\partial x} = -\frac{\partial h}{\partial y} \qquad (4)$$

$$\frac{\partial \Psi}{\partial y} = \frac{\partial h}{\partial x} \qquad (5)$$